Working Paper

FX Vanilla Pricing and Smile Calibration

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Abstract

The first part of this document is dedicated to the pricing of undiscounted Vanilla Options: we compute the price and sensitivities of Calls and Puts, and highlight the different FX Delta conventions.

The most important application of specific FX market conventions (and particularly Deltas conventions) is the FX smile construction. Indeed, the way the smile is described in forex market radically differs from other asset classes (equity, commodity...), requiring a specific procedure to recover the smile as a function of strike. In the second part of this document, we present two different approaches for the construction of one smile maturity slice. We apply both approaches to the calibration of UsdJpy 6 Months maturity smile and discuss the results.

keywords: Fx Conventions, Market Strangle, Fx Smile Calibration, SVI volatility

1 Introduction: Model and Conventions

Let denote X_{12}, one unit of *Currency1* quoted in *Currency2* (the risky asset is the *Currency1* measured in *Currency2*). We assume that X_{12} follows the Garman-Kohlhagen differential stochastic equation:

$$\frac{dX_{12}(t)}{X_{12}(t)} = \Delta\tilde{r}_{12}dt + \sigma_{12}dW_t$$

with $\Delta\tilde{r}_{12} = \tilde{r}_2 - \tilde{r}_1$, each $\tilde{r}_i$ denoting the swap rate and basis if the currency-i is not USD, or the swap rate only if currency-i is USD. This stochastic differential equation has a unique solution:

$$X_{12}(T) = X_{12}(t)\exp\left(\left(\Delta\tilde{r}_{12} - \frac{\sigma_{12}^2}{2}\right)(T-t) + \sigma_{12}(W_T - W_t)\right)$$

The Forward price The T-maturity forward price of X_{12} at time t, is given by:

$$F_{12}(t,T) = E[X_{12}(T)] = X_{12}(t)\exp(\Delta\tilde{r}_{12}(T-t))$$

The Forward value The forward with a maturity T, and a strike (K is the forward price K agreed upon at inception that makes the forward value zero), has the (undiscounted) value:

$$f_{12}(t,T) = F_{12}(t,T) - K$$

Actualization Factor Finally, we assume that the actualization factor $Act_2(t,T)$ in *Currency2* is independent from the forex dynamics, and applies as a multiplier to prices formula. Since $Act_2(t,T)$ depends on the collateral agreement of the trade (see [1]), we have decided to present all the results as undiscounted.

2 Vanilla Prices and Greeks

In this first part, we compute the Call and Put prices and their greeks.

2.1 Call and Put prices

With the same notations, *undiscounted* Call price C and Put price P are given by:

$$C(F_{12}(t),T,K,\sigma_{12}) = F_{12}(t,T)N(d_1) - KN(d_2)$$
$$P(F_{12}(t),T,K,\sigma_{12}) = KN(-d_2) - F_{12}(t,T)N(-d_1)$$

with

$$d_1 = \frac{1}{\sigma_{12}\sqrt{T-t}}\left(\ln\left(\frac{F_{12}(t,T)}{K}\right) + \frac{\sigma_{12}^2}{2}(T-t)\right)$$
$$d_2 = \frac{1}{\sigma_{12}\sqrt{T-t}}\left(\ln\left(\frac{F_{12}(t,T)}{K}\right) - \frac{\sigma_{12}^2}{2}(T-t)\right)$$

2.2 First Order Greeks

In this section we compute the first order sensitivities of option prices to their underlying parameters. Here again, we compute options Greeks for *undiscounted* options prices, and *discounted Greeks* can simply be deduced by multiplying *undiscounted* ones with $Act_2(t,T)$, the relevant actualization factor in *Currency2* .

Throughout this section, to compute Greeks, we will extensively use the symmetry relationship between $(F_{12}(t,T), n(d_1))$ in the one hand, and $(K,n(d_2))$ on the other hand:

$$F_{12}(t,T)n(d_1) = Kn(d_2)$$

2.2.1 Theta

When practitioners manage a portfolio of derivatives, the *Theta* is certainly - among all sensitivities - the one the more attention is paid to, because it clearly shows the breakeven of the *Theta-Gamma* PnL. The theoretical *Theta* for *undiscounted* Call and Put are given by:

$$\Theta_C = -\Delta\tilde{r}_{12}F_{12}(t,T)N(d_1) - Kn(d_2)\frac{\sigma_{12}}{2\sqrt{T-t}}$$
$$\Theta_P = Kn(d_2)\frac{\sigma_{12}}{2\sqrt{T-t}} + \Delta\tilde{r}_{12}F_{12}(t,T)N(-d_2)$$

This *Theta* calculation is mostly theoretical, and is not the one used in practice. Not only because our formula are not discounted and the discount factor $Act_2(t,T)$ does contribute to the *Theta*, but also because computing the impact of passage of time involves moving along curves (rate, term structure of volatility...) that are not flat, and have an impact on the result. For this reason, in practice, the *Theta* calculation must be performed under explicit assumptions (e.g. does the spot rate remain constant or does it follow the forward curve?...).

2.2.2 Deltas

The Delta is the quantity (the Delta is adimensional) of risky asset -i.e. *Currency1* - that must be held against an option in order to be hedged with respect to Forex spot moves.

But unlike options on equity, forex options can be settled either in *Currency2* or in *Currency1* (the risky asset), and in this last case, the Delta must be adjusted (we say *'Premium Adjusted Delta'*) to account for the premium. Then, the premium converted in *Currency1* must be substracted to the regular Delta (the premium comes as opposite to the regular Delta) to give the hedge amount.

On G10 currencies, the market convention distinguishes maturities above and below one year: below one year, options are hedged with spot Deltas, while above one forward Delta is used instead to simultaneously hedge swap point sensitivity. On emerging currencies, since swap points are more volatile, forward Delta hedge prevails for all maturities.

In this section, in addition to Call and Put Deltas, we compute the strike of the *Delta Neutral Straddle* (DNS), (the straddle with a strike such that the Call Delta + the Put Delta have a zero sum) and the Delta of its both legs.

2.2.3 Unadjusted Spot Delta

Undiscounted spot deltas of Call and Put are given by:

$$\Delta_S^C = \exp(\Delta\tilde{r}_{12}(T-t))N(d_1)$$
$$\Delta_S^P = -\exp(\Delta\tilde{r}_{12}(T-t))N(-d_1)$$

Discounted deltas are deduced from *undiscounted* ones by just multiplying them with the *Currency2* relevant discount factor $Act_2(t,T)$.

Call / Put Delta Parity The delta Call/Put parity of *undiscounted* vanilla options is

$$\begin{aligned}
\Delta_S^C - \Delta_S^P &= \frac{\partial C}{\partial X_{12}} - \frac{\partial P}{\partial X_{12}} \\
&= \exp\left(\Delta\tilde{r}_{12}(T-t)\right) N(d_1) - \exp\left(\Delta\tilde{r}_{12}(T-t)\right)(1 - N(d_1)) \\
&= \exp\left(\Delta\tilde{r}_{12}(T-t)\right)
\end{aligned}$$

DNS Strike and Legs Deltas

$$\frac{\partial C}{\partial X_{12}} + \frac{\partial P}{\partial X_{12}} = 0 \Rightarrow N(d_1) = N(-d_1) = \frac{1}{2}$$

It follows that $d_1 = 0$, and from d_1 definition, the ATM strike K_{ATM} must be:

$$\begin{aligned}
K_{ATM} &= X_{12}(t) \exp\left(\left(\Delta\tilde{r}_{12} + \frac{\sigma_{12}^2}{2}\right)(T-t)\right) \\
&= F_{12}(t,T) \exp\left(\frac{\sigma_{12}^2 (T-t)}{2}\right)
\end{aligned}$$

Deltas of *undiscounted* DNS Call and Put legs are given by:

$$\begin{aligned}
\Delta_S^{C\text{-ATM}} &= \frac{1}{2} \exp\left(\Delta\tilde{r}_{12}(T-t)\right) \\
\Delta_S^{P\text{-ATM}} &= -\frac{1}{2} \exp\left(\Delta\tilde{r}_{12}(T-t)\right)
\end{aligned}$$

and *discounted* ones are deduced by multiplying with $Act_2(t,T)$.

2.2.4 Unadjusted Forward Delta

Instead of spots, forward contracts can be used to hedge vanilla options. The Delta of *undiscounted* options with respect to *undiscounted* forward are given by:

$$\begin{aligned}
\Delta_F^C &= N(d_1) \\
\Delta_F^P &= -N(-d_1)
\end{aligned}$$

The formula for *discounted* options with respect to *discounted* forward is the same as the formula for *undiscounted* options with respect to *undiscounted* forward (presented above), as long as both options and forwards are managed under the same collateral agreement.

Call / Put Delta Parity The *undiscounted* forward Delta of *undiscounted* Call / Put parity (or equivalently the forward Delta of *discounted* Call / Put parity) is given by:

$$\Delta_F^C - \Delta_F^P = \frac{\partial C}{\partial f_{12}} - \frac{\partial P}{\partial f_{12}}$$
$$= N(d_1) + (1 - N(d_1)) = 1$$

DNS Strike and Legs Deltas

$$\frac{\partial C}{\partial f_{12}} + \frac{\partial P}{\partial f_{12}} = 0 \Rightarrow N(d_1) = N(-d_1) = \frac{1}{2}$$

It follows that $d_1 = 0$, and the ATM strike K_{ATM} must be:

$$K_{ATM} = X_{12}(t)\exp\left(\left(\Delta\tilde{r}_{12} + \frac{\sigma_{12}^2}{2}\right)(T-t)\right)$$
$$= F_{12}(t,T)\exp\left(\frac{\sigma_{12}^2(T-t)}{2}\right)$$

Undiscounted forward Deltas of *undiscounted* DNS (or equivalently, forward deltas of *discounted* DNS) Call and Put legs are given by:

$$\Delta_F^{C\text{-ATM}} = \frac{1}{2}$$
$$\Delta_F^{P\text{-ATM}} = -\frac{1}{2}$$

2.2.5 Premium Adjusted Spot Delta

If the *undiscounted* premium (C_{Prem} for the Call or P_{Prem} for the Put) is paid in risky currency, the spot Delta must be adjusted accordingly. The *undiscounted* spot Deltas of premium adjusted Call and Put are given by:

$$\Delta_{PS}^C = \frac{K}{X_{12}}N(d_2)$$
$$\Delta_{PS}^P = -\frac{K}{X_{12}}N(-d_2)$$

Call / Put Delta Parity *Undiscounted* Call / Put premium adjusted spot Delta parity is given by:

$$\Delta_{PS}^C - \Delta_{PS}^P = \frac{K}{X_{12}}N(d_2) + \frac{K}{X_{12}}(1 - N(d_2)) = \frac{K}{X_{12}}$$

DNS Strike and Legs Deltas

$$\frac{\partial C}{\partial X_{12}} - \frac{C}{X_{12}} + \frac{\partial P}{\partial X_{12}} - \frac{P}{X_{12}} = 0 \Rightarrow N(d_2) = N(-d_2) = \frac{1}{2}$$

It follows that $d_2 = 0$, and from d_2 definition, the ATM strike K_{ATM} must be:

$$\begin{aligned} K_{ATM} &= X_{12}(t)\exp\left(\left(\Delta\tilde{r}_{12} - \frac{\sigma_{12}^2}{2}\right)(T-t)\right) \\ &= F_{12}(t,T)\exp\left(-\frac{\sigma_{12}^2(T-t)}{2}\right) \end{aligned}$$

Premium adjusted spot Deltas of *undiscounted* DNS Call and Put legs are given by:

$$\begin{aligned} \Delta_{PS}^{C\text{-ATM}} &= \frac{1}{2}\frac{K_{ATM}}{X_{12}} = \frac{1}{2}\exp\left(\left(\Delta\tilde{r}_{12} - \frac{\sigma_{12}^2}{2}\right)(T-t)\right) \\ \Delta_{PS}^{P\text{-ATM}} &= -\frac{1}{2}\frac{K_{ATM}}{X_{12}} = -\frac{1}{2}\exp\left(\left(\Delta\tilde{r}_{12} - \frac{\sigma_{12}^2}{2}\right)(T-t)\right) \end{aligned}$$

2.2.6 Premium Adjusted Forward Delta

If the *undiscounted* premium (C_{Prem} for the Call or P_{Prem} for the Put) is paid in risky currency, the forward Delta must be adjusted accordingly. The *undiscounted* premium adjusted forward Deltas for Call and Put are given by:

$$\begin{aligned} \Delta_{PF}^{C} &= \frac{K}{F_{12}(t,T)}N(d_2) \\ \Delta_{PF}^{P} &= -\frac{K}{F_{12}(t,T)}N(-d_2) \end{aligned}$$

Here again, the formula for *discounted* options with respect to *discounted* forward is the same as the formula for *undiscounted* options with respect to *undiscounted* forward (presented above), as long as both options and forwards are managed under the same collateral agreement.

Call / Put Delta Parity The premium adjusted *undiscounted* forward Delta of *undiscounted* Call / Put parity (or equivalently the premium adjusted forward Delta of *discounted* Call / Put parity) is given by:

$$\Delta_{PF}^{C} - \Delta_{PF}^{P} = \frac{K}{F_{12}(t,T)}N(d_2) + \frac{K}{F_{12}(t,T)}(1 - N(d_2)) = \frac{K}{F_{12}(t,T)}$$

DNS Strike and Legs Deltas

$$\frac{\partial C}{\partial f_{12}} - \Delta_F^{\text{C-Prem}} + \frac{\partial P}{\partial f_{12}}\Delta_F^{\text{P-Prem}} = 0 \Rightarrow N(d_2) = N(-d_2) = \frac{1}{2}$$

It follows that $d_2 = 0$, and from d_2 definition, the ATM strike K_{ATM} must be:

$$\begin{aligned} K_{ATM} &= X_{12}(t)\exp\left(\left(\Delta\tilde{r}_{12} - \frac{\sigma_{12}^2}{2}\right)(T-t)\right) \\ &= F_{12}(t,T)\exp\left(-\frac{\sigma_{12}^2(T-t)}{2}\right) \end{aligned}$$

Undiscounted forward Deltas of *undiscounted* DNS (or equivalently, premium adjusted forward Deltas of *discounted* DNS) Call and Put legs are given by:

$$\begin{aligned} \Delta_{PF}^{C\text{-ATM}} &= \frac{1}{2}\frac{K_{ATM}}{F_{12}(t,T)} = \frac{1}{2}\exp\left(-\frac{\sigma_{12}^2(T-t)}{2}\right) \\ \Delta_{PF}^{P\text{-ATM}} &= -\frac{1}{2}\frac{K_{ATM}}{F_{12}(t,T)} = -\frac{1}{2}\exp\left(-\frac{\sigma_{12}^2(T-t)}{2}\right) \end{aligned}$$

2.2.7 Vega

The Call / Put parity ensures that call and put Vegas for a given strike are the same. The *undiscounted* Vega V is given by:

$$V = n(d_1)F_{12}(t,T)\sqrt{T-t} = n(d_2)K\sqrt{T-t}$$

2.3 Second Order Greeks

In this last section we compute (some) second order sensitivities. There is no market convention for such sensitivities because -unlike *Deltas* - there is no second order sensitivities exchanged at trade inception. In the following we dont compute all combinations of second order *Greeks* but we rather restrict our focus on sensitivities effectively used for risk management.

2.3.1 Gammas

When options are paid in *Currency1* (and Delta is premium adjusted), the Gamma must be adjusted as well.

Unadjusted Gamma From *unadjusted* Call / Put Delta parity we see that the *Unadjusted Gamma* of the put and the call are equal. The *undiscounted* Gamma is given by:

$$\Gamma = \exp(\Delta\tilde{r}_{12}(T-t))\,n(d_1)\,\frac{1}{X_{12}\sigma_{12}\sqrt{T-t}}$$

Premium Adjusted Gamma When the Delta is *premium adjusted*, the delta Call / Put parity shows that the premium adjusted Gamma of the Call and the Put differ: the premium adjustment is actually a convex function of the spot (the premium is converted back in *Currency1*), and as such, will contribute to the Gamma. Consequently, *undiscounted* premium adjusted Gamma for the Call and Put are different and are given by:

$$
\Gamma^{C} = \frac{F_{12}(t,T)}{X_{12}^{2}} \left(\frac{n(d_1)}{\sigma_{12}\sqrt{T-t}} + N(d_1) \right) + \frac{K}{X_{12}^{2}} N(d_2)
$$
$$
\Gamma^{P} = \frac{F_{12}(t,T)}{X_{12}^{2}} \left(\frac{n(d_1)}{\sigma_{12}\sqrt{T-t}} - N(-d_1) \right) + \frac{K}{X_{12}^{2}} N(-d_2)
$$

2.3.2 Vanna

The call put delta parity shows that the Vanna of a call is the same as the Vanna of a put with the same strike (the vega is the same for a call and a put with the same strike, so is their derivative with respect to the spot). The *undiscounted* value of the vanna can be expressed as a function of the *Vega*, and is given by :

$$
\text{Vanna} = -\exp\left(\Delta \tilde{r}_{12}(T-t)\right) n(d_1) \frac{d_2}{\sigma_{12}} = \frac{V}{X_{12}} \left(1 - \frac{d_1}{\sigma_{12}\sqrt{T-t}} \right)
$$

2.3.3 Volga

The Volga (*volatility gamma*) is the second derivative with respect to the volatility, or the derivative of the *Vega* with respect to the volatility. The *undiscounted* Volga is given by (V denotes the *undiscounted* vega) :

$$
\text{Volga} = n(d_1) F_{12}(t,T) \sqrt{T-t} \frac{d_1 d_2}{\sigma_{12}} = V \frac{d_1 d_2}{\sigma_{12}}
$$

Vanna-Volga management The Vanna-Volga method is a pricing method popularized among forex practitioners to price first generation exotic option in absence of a sophisticated volatility model. In a nutshell (see [2] for a full description), the main idea is to add to the Black-Scholes price of the exotic product, the cost of a portfolio of Risk-Reversal and Strangle options that matches both the Volga and Vanna sensitivities of the exotic product.

2.3.4 Delta Decay

The *Delta Decay* is the sensitivity of the Delta to passage of time:

$$
\text{DeltaDecay}^{C} = \frac{\partial C}{\partial X_{12} \partial t}
$$
$$
\text{DeltaDecay}^{P} = \frac{\partial P}{\partial X_{12} \partial t}
$$

The Delta Decay must be closely followed when managing a vanilla book, especially when options are close to their maturity: options Delta may vary a lot from one day to another (depending on how far the strike is from the spot) and requires some anticipation.

But just like the Theta computed above, the result is not really usable (because of rate curve, volatility curve...) as is. That is why we have decided that the *Delta Decay* computation will be left as an exercise for the reader.

3 Forex Smile Market Conventions

3.1 Forex Smile Description Format

For a given maturity, the volatility smile is commonly described by only five data:

σ_{ATM}	ATM volatility
$\Delta\sigma_{\text{RR25}}$	25% Delta Risk Reversal
$\Delta\sigma_{\text{MS25}}$	25% Delta Market Strangle
$\Delta\sigma_{\text{RR10}}$	10% Delta Risk Reversal
$\Delta\sigma_{\text{MS10}}$	10% Delta Market Strangle

- *ATM volatility* generally denotes for G10 currencies the *ATM zero delta*, i.e. the strike of a Delta Neutral Straddle (DNS). But for some emerging market, (e.g. South American currencies) ATM volatility denotes *At the money Forward*, that is, the volatility for a strike equal to the Forward price.
- The 25% Delta Risk Reversal (resp. 10% Delta Risk Reversal) is the difference of the 25% Delta Call (resp. 10% Delta Call) volatility and the -25% Delta Put (resp. -10% Delta Put) volatility.
- The 25% Delta Market Strangle defines with a unique spread both the price of Market Strangle, and the strike of each strangle leg (this point is further discussed in the next section).

The smile is not expressed as a function of the strike, and requires a specific construction procedure. In addition, the number of input data describing the smile is low, making such a procedure unstable.

3.2 Market Strangle and Smile Strangle

The *25% Market Strangle Volatility* is the volatility used to recompute the market price of the Market Strangle which is the sum of:

- a call that has a strike conventionally chosen such that its delta is 25% when priced with such *Market Strangle Volatility*
- a put that has a strike conventionally chosen such that its delta is -25% when priced with such *Market Strangle Volatility*

In other words, the 25% Market Strangle Volatility is not only used to price the Call and Put leg, but also defines the strike of the Call and the Put (evidently, the same convention applies to 10% Market Strangle Volatility). In Fx Market, Market Strangle Volatilities are specified by a spread $\Delta\sigma_{\text{MS25}}$ and $\Delta\sigma_{\text{MS10}}$ respectively against ATM volatility:

$$
\begin{aligned}
\sigma_{\text{MS25}} &= \sigma_{\text{ATM}} + \Delta\sigma_{\text{MS25}} \\
\sigma_{\text{MS10}} &= \sigma_{\text{ATM}} + \Delta\sigma_{\text{MS10}}
\end{aligned}
$$

Market Strangle prices are input for both calibration procedures presented below, and each will have to implement the following steps:

1. For a given delta (10% or 25% Delta) we solve the call strike and the put strike of the Market Strangle (possibly with a basic root search procedure),

2. From the strikes and the Market Strangle volatility, we deduce the Market Strangle prices. Those prices are input for our calibration procedures, for convenience we introduce MS_{25} and MS_{10} the 25% Delta Market Strangle price and 10% Delta Market Strangle price.

3. We compute the Market Strangle prices, but with different volatility for the Call and Put leg, each of them being read on the current smile subject to calibration: we denote MS^{Σ}_{25} and MS^{Σ}_{10} the 25% Delta Market Strangle and 10% Delta Market Strangle price found with our calibration procedure. The goal will be to minimize the distance of computed Market Strangle from input Market Strangles prices.

Unlike the Market Strangle, the *Smile Strangle* is a regular strangle whose marked-to-market price is the sum its Call and Put leg priced independently. Obviously the 25% Delta Smile Strangle will not have the same price, neither the same strikes, as the 25% Delta Market Strangle. We define 25% Delta Smile Strangle Spread and 10% Delta Smile Strangle Spread as:

$$\Delta\sigma_{\mathrm{SS25}} = \frac{1}{2}\left(\sigma^C_{25} + \sigma^P_{25}\right) - \sigma_{\mathrm{ATM}}$$
$$\Delta\sigma_{\mathrm{SS10}} = \frac{1}{2}\left(\sigma^C_{10} + \sigma^P_{10}\right) - \sigma_{\mathrm{ATM}}$$

where σ^C_{25} and σ^P_{25} (resp. σ^C_{10} and σ^P_{10}) are the 25% Delta (resp. 10% Delta) Call volatility, and -25% Delta (resp. -10% Delta) Put volatility.

We notice that, for a given Delta (25% or 10%), if we know both the Delta Smile Strangle Spread and the Delta Risk Reversal, we can algebraically deduce the value of the Delta Call volatility and the Delta Put volatility as follow:

$$\sigma^C_{25} = \sigma_{\mathrm{ATM}} + \frac{1}{2}\Delta\sigma_{\mathrm{RR25}} + \Delta\sigma_{\mathrm{SS25}} \tag{1}$$
$$\sigma^P_{25} = \sigma_{\mathrm{ATM}} - \frac{1}{2}\Delta\sigma_{\mathrm{RR25}} + \Delta\sigma_{\mathrm{SS25}} \tag{2}$$
$$\sigma^C_{10} = \sigma_{\mathrm{ATM}} + \frac{1}{2}\Delta\sigma_{\mathrm{RR10}} + \Delta\sigma_{\mathrm{SS10}} \tag{3}$$
$$\sigma^P_{10} = \sigma_{\mathrm{ATM}} - \frac{1}{2}\Delta\sigma_{\mathrm{RR10}} + \Delta\sigma_{\mathrm{SS10}} \tag{4}$$

This will be a key tenet for the first calibration approach.

4 Two Calibration methods for a Single Maturity Smile

In the following we present two methods to calibrate a maturity smile slice. The first approach is widely inspired by the paper from Dimitri Reiswich and Uwe Wystup [3], and results in calibrated volatility as a function of (Put) Delta.

The second approach uses the Jim Gatheral SVI [4] implied variance model and results in calibrated variance as a function of log-moneyness (it follows immediately an expression of volatility as a function of strike).

We show an example of calibration (6 months UsdJpy) and discuss the Pros and Cons of each method.

4.1 Delta to Volatility Mapping function

4.1.1 General principle

Dimitri Reiswich and Uwe Wystup in [3] calibrate a function that associates to the Put delta a Volatility:

$$\Sigma : \Delta^P \longmapsto \Sigma\left(\Delta^P\right)$$

Put Delta is preferred as parameter to Call Delta since it is a monotonous function of the strike, making the subsequent step (mapping strike to volatility) easier. We chose an Interpolator function Σ parametrized by five points (Δ_i^P, σ_i). In the next section, we define an objective function to minimize and ultimately construct the optimal mapping function Σ.

4.1.2 Description of calibrations steps

We construct an objective function d taking two Smile Strangles variables ($\Delta\sigma_{\mathrm{SS25}}$ and $\Delta\sigma_{\mathrm{SS10}}$) as arguments (all other parameters being provided as market quotes) and returning a distance (between prices) to be minimized.

1. Given the value of the 25% Delta Smile Strangle $\Delta\sigma_{\mathrm{SS25}}$ and 10% Delta Smile Strangle $\Delta\sigma_{\mathrm{SS10}}$, we deduce the volatility for five different (Put) Deltas. The first two points are Put volatilities from equations (2) and (4):

$$\Sigma(-25\%) = \sigma_{\mathrm{ATM}} - \frac{1}{2}\Delta\sigma_{\mathrm{RR25}} + \Delta\sigma_{\mathrm{SS25}} \tag{5}$$

$$\Sigma(-10\%) = \sigma_{\mathrm{ATM}} - \frac{1}{2}\Delta\sigma_{\mathrm{RR10}} + \Delta\sigma_{\mathrm{SS10}} \tag{6}$$

For the Call volatility we need to use the Call / Put Delta parity: $\Delta^C - \Delta^P = A$, where A depends on the delta convention (we have seen that A is not always constant and may depend on the spot and the strike), and possibly requires a root-finding algorithm to be defined. We denote A_{25} and A_{10}, the value of A for respectively the 25% and 10% Call / Put Delta parity, and rewrite (1) and (3), as function of put Delta:

$$\Sigma(25\% - A_{25}) = \sigma_{\mathrm{ATM}} + \frac{1}{2}\Delta\sigma_{\mathrm{RR25}} + \Delta\sigma_{\mathrm{SS25}} \tag{7}$$

$$\Sigma(10\% - A_{10}) = \sigma_{\mathrm{ATM}} + \frac{1}{2}\Delta\sigma_{\mathrm{RR10}} + \Delta\sigma_{\mathrm{SS10}} \tag{8}$$

The fifth point is actually given by the ATM Put Delta:

$$\Sigma(\Delta_{\mathrm{ATM}}) = \sigma_{\mathrm{ATM}}$$

Finally with five points $\left(\Delta_i^{\mathrm{P}}, \sigma_i\right)$ we can define our Interpolator function Σ. By construction, for all candidate function Σ, the Risk Reversal conditions are verified (from equations (7)-(5), and (8)-(6)):

$$\Sigma(25\% - A_{25}) - \Sigma(-25\%) = \Delta\sigma_{\mathrm{RR25}}$$
$$\Sigma(10\% - A_{10}) - \Sigma(-10\%) = \Delta\sigma_{\mathrm{RR10}}$$

2. We use our mapping function Σ to compute Market Strangles prices MS_{25}^{Σ} and MS_{10}^{Σ}. We know the Market Strangles strikes (see 3.2) but not pertaining Deltas, consequently a root-finding procedure is required once again.

3. From Market Strangles prices, we can compute a distance d to be minimized:

$$d\left(\Delta\sigma_{\mathrm{SS25}}, \Delta\sigma_{\mathrm{SS10}}; \sigma_{\mathrm{ATM}}, \Delta\sigma_{\mathrm{RR25}}, \Delta\sigma_{\mathrm{RR10}}\right) = \left|MS_{25} - MS_{25}^{\Sigma}\right|^2 + \left|MS_{10} - MS_{10}^{\Sigma}\right|^2$$

4.1.3 Remarks and Caveats

In [3] the study was restricted to a 3 points smile and the parametric function chosen was simply a second order polynomial. The extension to a 5 points smile is more complex. Since the optimization criteria involves the value of the functions Σ at interpolated (Put) Deltas (for the Market Strangle repricing), we must preclude Interpolator functions that possibly produces spurious oscillations between interpolated points (for instance we should avoid five points *natural Cubic Spline* Interpolator).

Beyond -10% Delta Put and 10% Delta Call, we must suggest an extrapolation solution that provides plausible shape for extrema of the curve.

4.2 SVI parametric volatility smile

4.2.1 General principle

In this section we present an overview of the SVI (*Stochastic Volatility Inspired*) smile geometry. For a given maturity, Jim Gatheral's model [4] describes implied variance with the parametric hyperbola arc:

$$var(y; a, b, \sigma, \rho, m) = a + b\left\{\rho(y - m) + \sqrt{(y - m)^2 + \sigma^2}\right\} \tag{9}$$

where the *log-moneyness* y is defined by $y = \ln\left(\frac{K}{F}\right)$, and $b \geq 0$, $|\rho| < 1$, $\sigma > 0$, and $a + b\sigma\sqrt{1 - \rho^2} > 0$ to ensure that the volatility remains positive for all log-moneyness. The SVI form has two asymptotes, that intersect at the point (m, a):

$$var_L(y; a, b, \sigma, \rho, m) = a - b(y - m)(1 - \rho)$$
$$var_R(y; a, b, \sigma, \rho, m) = a + b(y - m)(1 + \rho)$$

If $\rho < 0$ (resp. $\rho > 0$) the slope of left (resp. right) asymptote var_L (resp. var_R) is the steepest. Another important to define the geometry of our SVI smile is the location and the value of minimum implied variance. We exclude the $\rho = -1$ and $\rho = 1$ cases, where the volatility is strictly increasing or decreasing. We compute the first derivative of var with respect to the log-moneyness y:

$$\frac{\partial var}{\partial y} = b\left(\rho + \frac{y-m}{\sqrt{(y-m)^2+\sigma^2}}\right)$$

we deduce the log-moneyness $y_{\min}$ that minimizes the implied variance (we remark that $(y_{\min} < m)$ if $\rho > 0$ and $(y_{\min} > m)$ if $\rho < 0$) is given by:

$$y_{\min} = m - \frac{\rho\sigma}{\sqrt{1-\rho^2}}$$

The minimum value $var_{\min} = var\left(y_{\min}; a, b, \sigma, \rho, m\right)$ is:

$$var_{\min} = a + b\sigma\sqrt{1-\rho^2}$$

4.2.2 Description of calibrations steps

We introduce the volatility Σ as a function of log-moneyness:

$$\Sigma = \sqrt{\frac{var}{(T-t)}}$$

1. From the ATM strike (and corresponding log-moneyness) we get the volatility Σ_{ATM}, and we can compute a distance d^{Σ}_{ATM} :

$$d^{\Sigma}_{\text{ATM}} = \left|\sigma_{\text{ATM}} - \Sigma_{ATM}\right|^2$$

2. We don't have an immediate access to the Delta, so we use a root-finding algorithm and

   - solve the strike of the 25% Delta Call and its pertaining volatility Σ_{C25},
   - solve the strike of the -25% Delta Put and its pertaining volatility Σ_{P25},
   - solve the strike of the 10% Delta Call and its pertaining volatility Σ_{C10},
   - solve the strike of the -10% Delta Call and its pertaining volatility Σ_{P10},

   we can then compute 25% and 10% Risk Reversals:

$$\Delta\sigma^{\Sigma}_{\text{RR25}} = \Sigma_{\text{C25}} - \Sigma_{\text{P25}}$$
$$\Delta\sigma^{\Sigma}_{\text{RR10}} = \Sigma_{\text{C10}} - \Sigma_{\text{P10}}$$

   and define a distance d^{Σ}_{RR}

$$d^{\Sigma}_{\text{RR}} = \left|\Delta\sigma_{\text{RR25}} - \Delta\sigma^{\Sigma}_{\text{RR25}}\right|^2 + \left|\Delta\sigma_{\text{RR10}} - \Delta\sigma^{\Sigma}_{\text{RR10}}\right|^2$$

3. We use our parametric smile Σ to compute Market Strangles prices MS^{Σ}_{25} and MS^{Σ}_{10}. We define a distance d^{Σ}_{MS}:

$$d^{\Sigma}_{\text{MS}} = \left|MS_{25} - MS^{\Sigma}_{25}\right|^2 + \left|MS_{10} - MS^{\Sigma}_{10}\right|^2$$

Finally, the distance d^{Σ} to be minimized is:

$$d^{\Sigma} = d^{\Sigma}_{\text{MS}} + \text{vega}^2_{\text{ATM}}\, d^{\Sigma}_{\text{ATM}} + \text{vega}^2_{\text{RR}}\, d^{\Sigma}_{\text{RR}}$$

where weights vega_{ATM} and vega_{RR} for price homogeneity purpose.

4.2.3 Remarks and Caveats

We can observe empirically the impact of each parameter (see [4]):

- a gives the overall level of variance
- b gives the angle between the left and right asymptotes
- σ determines how smooth the vertex is
- ρ determines the orientation of the graph
- m translates the smile left- or rightwards

This creates an issue since different parameters can have similar effects on the Forex smile, for instance a shift of m will have an important impact on the risk reversal, and so does b and obviously *rho*. The fact that parameters don't have independent effects on the smile generally makes optimization more complicated and the solution of calibration not unique: multiple sets of optimal parameters can be found to calibrate the smile. In addition (and partially for the same reason), the minimization procedure shows a high dependency on initial guesses of parameters.

Strengthening and possibly adding additional constraints is a good way to circumvent these issues.

4.3 Calibration Example: 6 Months UsdJpy

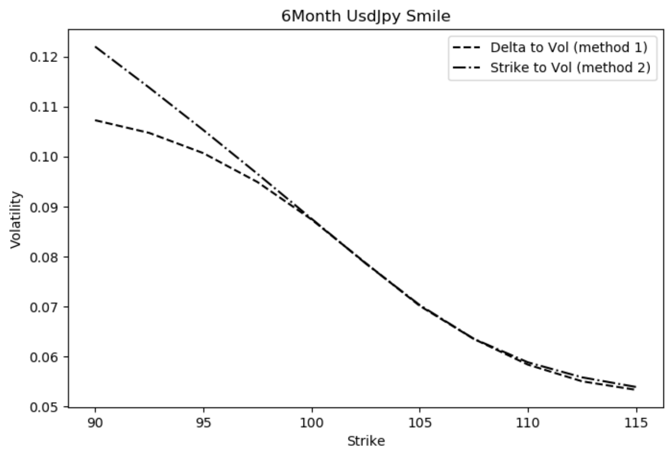
We calibrate a 6months smile slice of UsdJpy. The market data related to UsdJpy forex rate are:

- UsdJpy spot $X_{12} = 105.28$
- UsdJpy 6 months outright $F_{12} = 106.6985$
- Jpy 6 months actualization factor $Act_2 = 1.00017$

The 6months smile is described by the 5 parameters:

σ_{ATM}	0.0658
$\Delta\sigma_{\text{RR25}}$	-0.019
$\Delta\sigma_{\text{MS25}}$	0.00175
$\Delta\sigma_{\text{RR10}}$	-0.038
$\Delta\sigma_{\text{MS10}}$	0.00853

We use the two calibration methods and we get the following smiles:



Results comparison In practice, SVI volatility is more complicated to calibrate, and requires a close attention to the 'average' smile geometry to define initial guesses parameters. Nevertheless, once calibrated, the SVI implied variance is much more convenient to use, because it can be easily expressed as a function of the strike, whereas the first approach requires the strike to be converted in delta to access the volatility and this conversion is performed with a root-search procedure.

Obviously, the volatility curve extrema are more realistic with the SVI form than the first method. Despite the use of linear extrapolation for the mapping function between Delta and volatility, the first method shows a volatility flattening for low Delta Put. We know that this method cannot deal with extreme strikes: for extreme strikes, the Delta as a function of strike is hardly bijective, making the final mapping between strike and volatility more error prone. Clearly, for the first method to be fully reliable, a workaround for extrapolated volatilities must be further studied. In contrast, the SVI hyperbola arc offers a *'built in'* asymptotic behavior for low delta volatility.

Applications and Next steps The main application of one maturity volatility slice calibration is obviously the calibration of the volatility smile across different maturities. Each maturity slice can be constructed independently (while including a way to prevent butterfly arbitrage), and the main process will gather all maturity slices and ensure absence of calendar arbitrage.

Another application is the creation of smile time series for a single maturity slice. Indeed, we have seen that the low number of data describing the Forex smile makes the smile construction more difficult, but conversely it makes the handling of smile time series easier. From time series of smile parameters we can construct time series of calibrated smile slices and extract analytics to build other time series: for example, from one maturity calibrated smile we can easily compute the fair variance swap strike. This is prerequisite to analyze time series of variance swaps (for instance with the language R or Python's packages `scipy.stats` and `statsmodel`) and construct relevant trading strategies.

I thank Nicolas Rolland for useful remarks and comments. All remaining errors are mine.

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